Derivative of a Composite Function

EXAMPLE 1 Relating Derivatives

The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

$$\frac{dy}{dx} = 6 \qquad \frac{dy}{du} = 2 \qquad \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{\text{outside function}} \cdot \underbrace{\frac{du}{dx}}_{\text{inside function}}$$

EXAMPLE 2 Relating Derivatives

The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

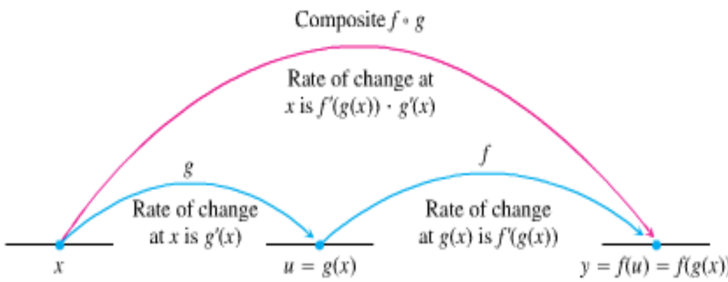
$$\frac{dy}{dx} = 36x^3 + 12x \qquad \frac{dy}{du} = 2u = 2(3x^2+1) = 6x^2+2 \qquad \frac{du}{dx} = 6x$$

$$\frac{dy}{du} \cdot \frac{du}{dx} = 6x(6x^2+2) = 36x^3 + 12x$$

$$f(x) = \underbrace{(g(x))^2}_{\text{outside}} \qquad g(x) = \underbrace{3x^2+1}_{\text{inside}} \qquad \underbrace{f'(x) = 2(g(x)) \qquad g'(x) = 6x}$$

$$f'(g(x)) \cdot g'(x)$$

Derivative of outside evaluated at the inside times the derivative of the inside.



RULE 8 The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

$f(\cancel{t}) = \cos(g(t))$

$g(\cancel{t}) = t^2 + 1$

$$v(t) = x'(t) = f'(g(t)) \cdot g'(t)$$

$$= -\sin(t^2+1) \cdot (2t)$$

$$= -2t \sin(t^2+1)$$

"Outside-Inside" Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

EXAMPLE 4 Differentiating from the Outside in

Differentiate $\sin(x^2 + x)$ with respect to x.

$f(x) = \sin(g(x))$

$g(x) = x^2 + x$

SOLUTION

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\text{inside left alone}} \cdot \underbrace{(2x + 1)}_{\text{derivative of the inside}}$$

Repeated Use of the Chain Rule

EXAMPLE 5 **A Three-Link "Chain"**

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

$f(t) = \tan(g(t))$

$g(t) = 5 - \sin 2t$

$= f'(g(t)) \cdot g'(t)$

$= \sec^2(5 - \sin 2t) \cdot (-2\cos 2t)$

$= -2\cos 2t \sec^2(5 - \sin 2t)$

$f(t) = 5 - \sin(g(t))$

$g(t) = 2t$

$f(g(t)) \cdot g'(t)$

$-\cos 2t \cdot 2$

Power Chain Rule

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}.$$

Here's an example of how it works: If n is an integer and $f(u) = u^n$, the Power Rules (Rules 2 and 7) tell us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule:**

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 7 Finding Slope

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

a) $y = (\sin x)^5$

$y' = 5(\sin x)^4 \cdot \cos x$

$= 5 \sin^4 x \cos x$

$y'(\pi/3) = 5\left(\sin \frac{\pi}{3}\right)^4 \cos \frac{\pi}{3}$

$\left(\left(\frac{\sqrt{3}}{2}\right)^2\right)^2$

$= 5\left(\frac{\sqrt{3}}{2}\right)^4 \cdot \frac{1}{2}$

$= 5 \cdot \frac{9}{16} \cdot \frac{1}{2} = \frac{45}{32}$

b) $\frac{1}{(1-2x)^3} = (1-2x)^{-3}$

$y' = -3(1-2x)^{-4} \cdot (-2)$

$= \frac{6}{(1-2x)^4}$

6 is positive, anything to the fourth power is always positive and positive divided by positive is positive.

EXAMPLE 8 Radians Versus Degrees

It is important to remember that the formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees.